

# Voter Models and External Influence

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## Abstract

In this paper, we extend the voter model (VM) and the threshold voter model (TVM) to include external influences modelled as a jump process. We study the newly-formulated models both analytically and computationally, employing diffusion approximations and mean field approximations. We derive results pertaining to the probability of reaching consensus on a particular opinion and also the expected consensus time. We find that although including an external influence leads to a faster consensus in general, this effect is more pronounced in the VM as compared to the TVM. Our findings suggest the potential importance of “macro-level” phenomena such as the external influences as compared to “micro-level” local interactions.

*Keywords* - voter model, threshold voter model, opinion dynamics, external influence, mean field approximation, diffusion approximation.

## 1 Introduction

Human opinions and collective human behaviours have been studied for more than a century.<sup>1, 17, 20, 22, 23, 27, 28</sup> Relatively recent is the mathematical analyses of these dynamics, which have been made possible due to seminal frameworks such as the voter model (VM),<sup>8, 16</sup> DeGroot learning,<sup>12</sup> and the naming game model.<sup>3, 10</sup> A significant number of advances in opinion dynamics have their roots in statistical physics.<sup>5, 24, 30</sup> For details on the origins and evolution of this domain, the reader is referred to the comprehensive review by Castellano et al.<sup>7</sup>

While refining the mechanistic description of node-to-node interactions has received considerable attention,<sup>9, 15, 21</sup> the incorporation of external influences has received little modelling scrutiny, one reason for which seems to be the associated loss of analytical tractability. There is evidence, however, that media can play an important role in opinion dynamics in many different contexts, for example, in climate change,<sup>6</sup> and in electoral voting.<sup>4, 13</sup> Furthermore, commercial relevance of this phenomenon can be found in quantitative marketing, when multiple brands compete to sell their respective products via advertising. A theoretical examination of the effect of external influence is thus important. Existing work, however, is either not amenable to detailed mathematical analysis,<sup>26</sup> or lacks generality.<sup>14</sup> We address these shortcomings by modelling external influence as a jump process, and use the theory of jump diffusion processes. Jump diffusion models are widely used in financial domains such as derivative pricing and risk management.<sup>18</sup>

The organisation of the paper is as follows: In section 2 we define our models, followed by our results and their interpretation in section 3. We finish with a high-level discussion in section 4.

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## 2 Models

Let  $G$  be a  $k$ -regular undirected graph of  $N$  nodes, where each node is considered to be an individual agent and the links represent social connections between the agents. Let  $S(G)$  denote the set of nodes of  $G$ . Consider a binary opinion space, i.e., a node  $x \in S(G)$  must have either opinion 0 or opinion 1 at time  $t$  which is denoted by  $s_x(t)$ . Let  $A$  be the adjacency matrix, with the element  $A_{xy}$  being 1 if nodes  $x$  and  $y$  are connected, and 0 otherwise. The neighbourhood of a node consists of the immediate nodes with which it is connected. *Consensus* is said to be reached when either all nodes have opinion 0 or opinion 1, and it is assumed that these states are absorbing. Consensus on opinion 1 is herein called *fixation*.

### 2.1 Jump Voter Model

The jump voter model (JVM) is a discrete-time process that is updated according to two rules. At each time step, one of the following occurs:

1. With probability  $(1 - p)$ , a single node is randomly selected which then adopts the opinion of one of its neighbours chosen randomly.
2. With probability  $p$ , either a random number of 0 opinion nodes update their opinions to 1 or a random number of 1 opinion nodes update their opinions to 0. If this number exceeds the number of nodes available to be updated, then the all the available nodes are updated. A signed form of this random variable ( $Z$ ) follows the convention that it is negative in the case of the former update, and positive for the latter.

The first update rule captures the node-to-node interactions of the “classical” VM. The second rule captures the more global external influence that makes several opinions flip simultaneously, a phenomenon that we call a jump. Call  $p$  the jump probability, and note that when  $p = 0$  the model reduces to a discrete-time version of the VM on a graph structure. The JVM is a discrete state space Markov chain, and the total number of opinion 1 nodes in the graph at time step  $t$ , denoted by  $X^N(t)$ , can be thought of as a global summary statistic of that Markov chain. More formally,

$$X^N(t) = \sum_{x \in S(G)} s_x(t).$$

The jump random variable  $Z$  is independent of the state of the process, and we define the scaled jump  $Y \equiv Z/N$ . We restrict our attention to the case where the mean of  $Y$  is zero, that is the case of no bias in the external influence. The variance of  $Y$ , denoted by  $v$ , represents the strength of the external influence, and we call it the jump variance. (A brief summary of all the model parameters is provided in Table 1.)

### 2.2 Jump Voter Diffusion

We make use of a diffusion approximation here and derive a jump-diffusion process to which the JVM weakly converges. To proceed with this jump diffusion approximation, we will need the transition probabilities corresponding to update rule 1 (i.e.  $p = 0$ ):

- $P[i \rightarrow i + 1] \equiv$  probability that a  $0 \rightarrow 1$  update happens at a certain time step when the count of nodes with opinion 1 is  $i$ , and,

- $P[i \rightarrow i-1] \equiv$  probability that a  $1 \rightarrow 0$  update happens at a certain time step when the count of nodes with opinion 1 is  $i$ .

Therefore, we have,

$$\begin{aligned} P[i \rightarrow i+1] &= \sum_{\substack{x \in S(G), \\ s_x=0}} \left( \frac{1}{N} \sum_{y \in S(G)} A_{xy} \frac{s_y}{k} \right) \\ P[i \rightarrow i-1] &= \sum_{\substack{x \in S(G), \\ s_x=1}} \left( \frac{1}{N} \sum_{y \in S(G)} A_{xy} \frac{1-s_y}{k} \right) \end{aligned} \quad (1)$$

Using a mean-field approximation, in the spirit of that used by Sood et al,<sup>29</sup> the transition probabilities in equation (1) become,

$$\begin{aligned} P[i \rightarrow i+1] &\approx \left(1 - \frac{i}{N}\right) \left(\frac{i}{N}\right) \\ P[i \rightarrow i-1] &\approx \left(\frac{i}{N}\right) \left(1 - \frac{i}{N}\right). \end{aligned} \quad (2)$$

For large  $N$ , the scaled process corresponding to the update rule 1,  $X^N[N^2t]/N$ , converges to a diffusion  $X(t)$  whose drift and diffusion terms derived using the transition probabilities from equation (2) are,

$$\begin{aligned} \mu(x) &= 0 \\ \sigma^2(x) &= 2x(1-x). \end{aligned} \quad (3)$$

Additionally, if we define  $\lambda \equiv N^2p$  (scaled jump), then for a small enough  $p$  and a large enough  $N$ , the JVM (update rules 1 and 2) can be approximated by a superposition of the diffusion derived above and a compound Poisson process, i.e., a jump diffusion process given as,

$$dX(t) = \sqrt{2X(t)(1-X(t))}dW(t) + YdN(t) \quad (4)$$

where  $W(t)$  represents a Wiener process, and  $N(t)$  represents a rate  $\lambda$  Poisson process. We denote by  $g(x)$  the probability density function of the jump random variable  $Y$ , and also note that  $g(x)$  will have support  $[-1, 1]$ . The generator of this process is the integro-differential operator  $\mathcal{L}$ , defined by

$$\mathcal{L}f(x) = x(1-x)f''(x) + \lambda \int_{-\infty}^{+\infty} [f(x-y) - f(x)]g(y)dy. \quad (5)$$

Note here that as a result of using the mean field approximation, the jump voter diffusion given by equation (4) does not have a term containing the degree parameter  $k$ .

### 2.3 Jump Threshold Voter Model

The jump threshold voter model (JTVM) is a discrete-time process that is updated according to two rules. At each time step, one of the following occurs:

1. With probability  $(1 - p)$ , a single node is randomly selected. If the number of opposing opinions in neighbourhood of the selected node is greater than or equal to a threshold  $\theta$ , then the opinion of the originally selected node is updated.
2. With probability  $p$ , either a random number of 0 opinion nodes update their opinions to 1 or a random number of 1 opinion nodes update their opinions to 0. If this number exceeds the number of nodes available to be updated, then the all the available nodes are updated. A signed form of this random variable ( $Z$ ) follows the convention that it is negative in the case of the former update, and positive for the latter.

As with the JVM, the mean of  $Z$  is zero herein.

Table 1 summarises all the parameters discussed in this section.

**Table 1:** Summary of the model parameters.

Parameter	Brief Description
$N$	Total population size.
$k$	Degree in a regular graph.
$p$	Jump probability at each time step.
$Z$ (or $Y$ )	Random variable that gives the jump or the external influence.
$v$	Jump variance, or strength of the external influence.
$\theta$	Threshold parameter in the JTVM.
$\lambda$	Jump rate in the jump voter diffusion.

### 3 Results

If  $X(t)$  is a jump diffusion with generator  $\mathcal{L}$ , Itô's formula for jump processes<sup>25</sup> implies that

$$M(t) \equiv f(X(t)) - \int_0^t \mathcal{L}f(X(s))ds \quad (6)$$

is a martingale for any  $C^2$  function  $f(x)$ . Application of the Optional Stopping Theorem (OST) to this martingale results in boundary value problems for both fixation probability and expected value of the consensus time for the jump voter diffusion. Before that, we mathematically define the first hitting times of the two absorbing states (0 and 1) and the consensus time ( $\tau$ )

$$\begin{aligned} T_0 &\equiv \inf\{t \geq 0 : X(t) \in (-\infty, 0]\}, \\ T_1 &\equiv \inf\{t \geq 0 : X(t) \in [1, \infty)\}, \\ \tau &\equiv \inf\{t \geq 0 : X(t) \in (-\infty, 0] \cup [1, \infty)\}. \end{aligned}$$

The fixation probability  $u(x) = P(T_1 < T_0 | X(0) = x)$  satisfies,

$$\begin{aligned} \mathcal{L}u(x) &= 0, \\ u(x) &= 0, \quad \forall x \in (-\infty, 0], \\ u(x) &= 1, \quad \forall x \in [1, \infty). \end{aligned} \quad (7)$$

Similarly, expected value of the consensus time  $v(x) = E[\tau|X(0) = x]$  satisfies,

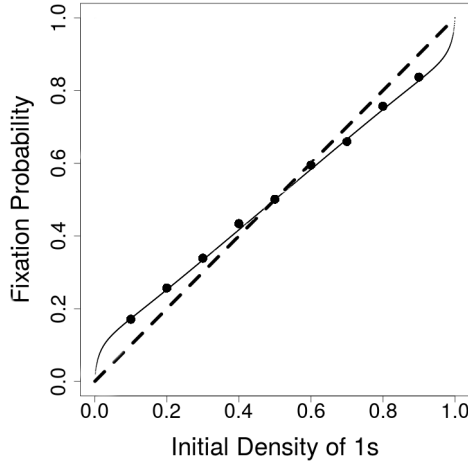
$$\begin{aligned}\mathcal{L}v(x) &= -1, \\ v(x) &= 0, \forall x \in (-\infty, 0] \cup [1, \infty).\end{aligned}\tag{8}$$

### 3.1 Jump Voter Model

We begin by investigating the fixation probability and the expected consensus time for the jump voter diffusion. To determine the fixation probability, we use the generator of the jump voter diffusion from equation (5) in the BVP (7), to obtain,

$$\begin{aligned}x(1-x)u''(x) + \lambda \int_{-\infty}^{+\infty} u(x-y)g(y)dy - \lambda u(x) &= 0 \\ u(x) &= 0, \forall x \in (-\infty, 0] \\ u(x) &= 1, \forall x \in [1, \infty).\end{aligned}\tag{9}$$

It is quite challenging to find an analytical solution for a variable-coefficient integro-differential equation such as equation (9). Even a similar constant-coefficient equation requires imposing some structure on the function  $g(x)$  to make a closed-form solution possible.<sup>19</sup> We thus solve this problem numerically and compare the solution to Monte Carlo simulations of the JVM (Figure 1).

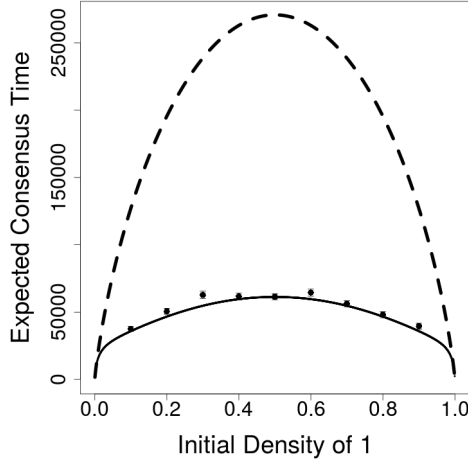


**Figure 1:** Fixation probability for the JVM on a regular graph with  $N = 500, k = 4, p = 1/(500 \times 10), v = 0.04$  (see Table 1 for meaning of parameters). (---) denotes the solution for the VM, (—) denotes the numerical solution of equation (9), (●) denotes simulation results based on update rules in Section 2.1, where each point is obtained by averaging over 1000 runs. The external influence,  $Z$ , has a truncated normal distribution.

Similarly, if we expand equation (8) using the generator in equation (5), we obtain

$$\begin{aligned}x(1-x)v''(x) + \lambda \int_{-\infty}^{+\infty} v(x-y)g(y)dy - \lambda v(x) &= -1 \\ v(x) &= 0, \forall x \in (-\infty, 0] \cup [1, \infty).\end{aligned}\tag{10}$$

Solving the problem numerically, we find that the solution again closely matches the results of the Monte Carlo simulations of the JVM (Figure 2).



**Figure 2:** Expected consensus time for the JVM on a  $25 \times 25$  lattice, with  $p = 1/(625 \times 10)$ ,  $v = 0.03$  (see Table 1 for meaning of parameters). (---) denotes the solution for the VM, (—) denotes the numerical solution of equation (10), (•) denotes simulations results based on update rules in Section 2.1, where each point is obtained by averaging over 500 runs, and error bars indicate standard error of the mean. The external influence,  $Z$ , has a truncated normal distribution.

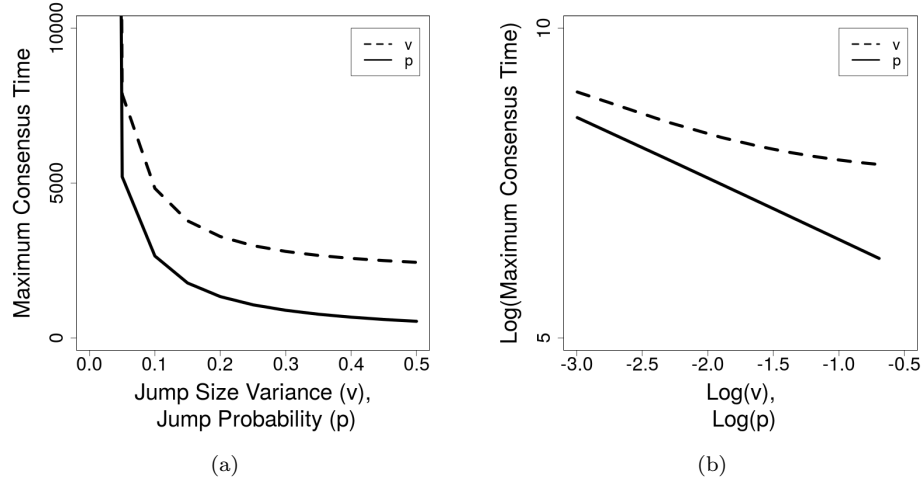
We notice in Figure 2 that although our solution for the consensus time has properties that are qualitatively similar to those of the VM, the quantitative difference between the two is considerable. The two parameters, jump probability  $p$  and jump variance  $v$ , together determine the overall impact of the external influence, and we collectively refer to them as jump parameters. We find that even for fairly small jump parameter values, the expected consensus time differs dramatically between the VM and the JVM.

If we consider just the maximum value of consensus time (which occurs at initial density of 1s is 0.5), we can plot it as a function of the two jump parameters (Figure 3). We see that the consensus time decreases rapidly as a function of  $p$  and  $v$ , especially at small values.

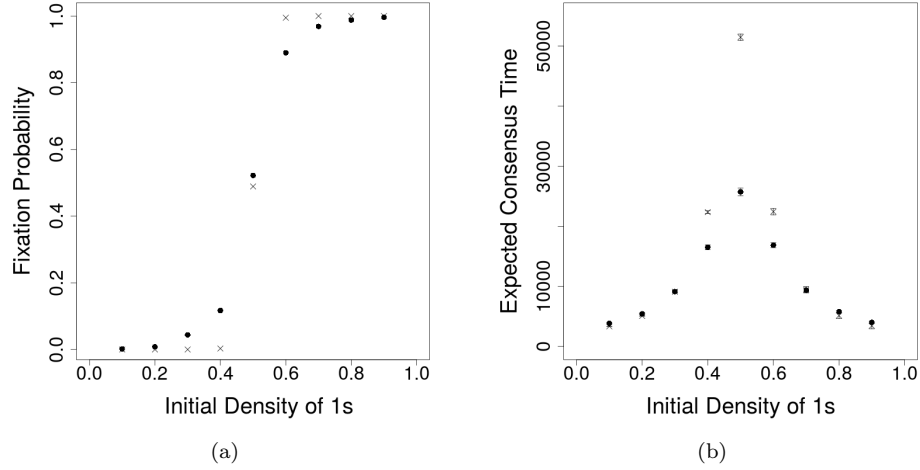
*Consensus time in the VM is sensitive to the presence of jumps.* Consensus time decreases in both jump parameters ( $p$  and  $v$ ), and the rate of decrease of consensus time is very high at low jump parameter values, and significantly drops as parameter values increase (Figure 3(a)). Therefore, it is primarily the presence of jumps that appears to be a key factor for the consensus time. Consistent with that observation, the dependence of consensus time on both jump parameters exhibits a power-law like nature (Figure 3(b)). Overall, jumps expedite consensus, introducing little skew in addition to that inherently present due to the initial densities. Therefore, jumps may be an important ingredient to consider in opinion dynamics models based on the VM.

### 3.2 Jump Threshold Voter Model

In this section, we study the fixation probability and consensus time for the JTVM. The results are obtained through Monte Carlo simulations, and are shown in Figure 4.

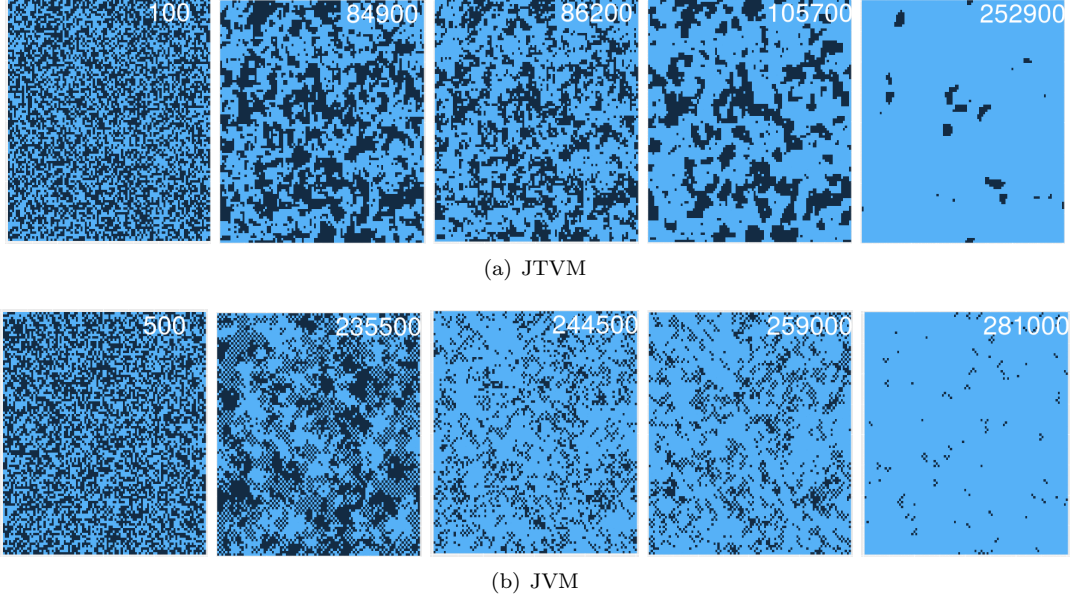


**Figure 3:** (a) Maximum consensus time for the jump voter diffusion as a function of the jump parameters individually,  $N = 500$ . (The maximum consensus time axis has been restricted to 10000 time steps for diagrammatic clarity.) The maximum consensus time decreases rapidly as  $p$  and  $v$  increase. For plotting the dependence on  $p$ ,  $v = 0.001$ , and for plotting the dependence on  $v$ ,  $p = 0.001$  (see Table 1 for meaning of parameters). Plot (b) is plot (a) on a log-log scale.



**Figure 4:** (a) Fixation probability and (b) expected consensus time for the JTVM ( $\bullet$ ) in comparison with the TVM ( $\times$ ), on a  $25 \times 25$  lattice with  $p = 1/(625 \times 10)$ ,  $v = 0.03$ ,  $\theta = 2$  (see Table 1 for meaning of parameters). Each point is based on 1000 runs, and error bars indicate standard error of the mean. The external influence,  $Z$ , has a truncated normal distribution.

The spatial structure of the dynamics yields further insight into the model behaviour. Typical spatial results are shown in Figure 5. It is known that the evolution of clusters in the TVM is characterised by motion by mean curvature.<sup>7,11</sup> The introduction of jumps to this model plays the role of disrupting the clustering sporadically. This can be seen in Figure 5(a).



**Figure 5:** Snapshots of evolution of (a) the JTVM, and (b) the JVM on a  $100 \times 100$  lattice, with  $p = 1/(10000 \times 5)$ ,  $v = 0.03$ ,  $\theta = 2$  (see Table 1 for meaning of parameters). Initial density of 1s is 0.5 for both models. The numbers in the top right corner denote the time step for the respective panel. The external influence,  $Z$ , has a truncated normal distribution.

The JTVM is similar, for the fixation probability and the consensus time, to the TVM at low initial minority densities. The differences between the TVM and the JTVM become prominent only at initial minority densities greater than 0.4 approximately, which we call the “critical” density. The fixation probability for the TVM exhibits behaviour similar to a step function (Figure 4), where consensus is almost always reached on the opinion in majority at the start of the process. The TVM thus amplifies the advantage held by to a particular opinion type due to a higher initial density. Comparing the fixation probability in this case to that for the JTVM, we notice that the probabilities become less extreme at initial minority densities greater than the critical density. Therefore, at initial minority densities greater than the critical density, jumps help moderate the advantage amplification inherent to the TVM. Next we assess the effect of jumps on the consensus time. Jumps have an effect of increasing fluctuations in the opinion density. This effect appears more pronounced at initial minority densities higher than the critical density, as we obtain a quicker consensus in that regime.

Overall, for initial minority densities lower than the critical density, the clustering effect is too strong to be disrupted by jumps. However, once the initial minority density exceeds a certain value, the disrupting effects of jumps begin to counter the clustering effect inherent to the TVM.

*Jumps expedite consensus in the TVM less than the VM.* We notice, based on comparing Figures 2 and 4, that jumps reduce the consensus time significantly more in the VM than in the TVM. This suggests that the TVM is more robust to an external influence. This is again attributable to motion by mean curvature in the TVM. Consider the extreme case, for both the JVM and JTVM, where the jump causes a  $0 \rightarrow 1$  flip at a node in the interior of a 0 cluster. In the JVM, there is a non-zero probability that the 1 in the interior can in turn cause a  $0 \rightarrow 1$  flip at one of its neighbouring nodes. But in the TVM, the probability that this opinion 1 node can flip any of its opinion 0 neighbours is zero. Therefore, the cluster patterns of the VM tend to be more vulnerable to jumps as compared to those of the TVM. This resistance serves as a possible explanation for the higher robustness of the TVM to jumps with regards to the consensus time. This phenomenon can also be observed in Figure 5: Notice the similarity in the cluster pattern in the second and fourth panels of Figure 5(a) despite the occurrence of a jump in the third panel. On the contrary, no such trend is seen in Figure 5(b).



## 4 Discussion

In this work, we have developed extensions of the voter model (VM) and the threshold voter model (TVM), that incorporate an external influence in the form of many opinions shifting in the same direction simultaneously. Most existing literature on opinion dynamics only studies opinion evolution under influences internal to the system. This work provides a systematic study of opinion dynamics under an additional external influence. We approximated the jump voter model (JVM) by means of a jump diffusion process. This approach allowed us to analytically determine the probability of reaching consensus on opinion 1 (fixation probability) and the mean consensus time. For the jump threshold voter model (JTVM), we relied on simulations to determine fixation probability and mean consensus time.

The diffusion approximation for the VM does not depend on the degree of the regular graph,<sup>29</sup> and we note that this property is retained in the JVM. This means that the fixation probability and consensus time results for the JVM will remain true for all regular graphs ranging from a complete graph (where  $k = N - 1$ ) to even a cycle graph (where  $k = 2$ ). Thus, in a society where everyone has the same neighbourhood size, fixation probability and consensus time are independent of the neighbourhood size as long as agents update their opinions by a combination of random sampling from their neighbourhood and by an unbiased external influence.

Another key observation is that jumps expedite consensus more in the VM than in the TVM. Thus, in a society where agents update their opinion if the pressure from their neighbourhood is sufficient (based on a threshold parameter), external influence has a lesser effect on consensus time than in a society where agents update their opinions by randomly sampling from their neighbourhood.

This work opens up multiple interesting directions that may be further explored. The domain of network science is currently expanding very rapidly, and one natural extension of our work is to study our models on a heterogeneous graph structure such as a scale-free network. Such work could lead to pragmatic insights since scale-free networks have been shown to be ubiquitous in various real-world social systems.<sup>2</sup>

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